**Average Value of a Function**

The table below shows a record of the velocity of a car in miles per hour every 5 seconds over a period of 50 seconds:

| $Time(s)$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Velocity(v(t))$ | 0 | 20 | 30 | 45 | 50 | 60 | 55 | 50 | 60 | 65 | 70 |

To find the average speed of the car in the 50 second time period, we could use the average of the data given above (omitting the 0 mph when $t = 0$)

$$\frac{20 + 30 + 45 + 50 + 60 + 55 + 50 + 60 + 65 + 70}{10} =$$

However these numbers do not take into consideration the velocity of the car at the many points in the 5 second intervals between the observations and the velocity may vary on these intervals. Since velocity is a continuous function, we do not expect it to vary wildly on a small interval. To get a better estimate of the average velocity we could take observations over smaller time intervals with length $\Delta x = \frac{50-0}{n}$ and compute the average of the collected data

$$\frac{1}{n}\sum_{i=1}^{n} v(x_i) = \frac{1}{50}\frac{50}{n}\sum_{i=1}^{n} v(x_i) = \frac{1}{50}\sum_{i=1}^{n} v(x_i)\Delta x.$$

In fact we define the average velocity of the car over the time interval $[0, 50]$ to be the limit of this sum as the time intervals approach zero. Thus the average velocity of the car over the 50 second period is given by

$$v_{ave} = \lim_{\Delta x \to 0} \frac{1}{50}\sum_{i=1}^{n} v(x_i)\Delta x = \frac{1}{50}\int_0^{50} v(t)dt.$$

(Note if we do not have a formula for $v(t)$ or a way to find an antiderivative for $v(t)$, the best way to calculate the velocity is to use data with as many observations as possible as above.)

**Note** we might also calculate the average velocity of the car on the time interval $[0, 50]$ by calculating the change in displacement and dividing by 50. Why does this give us the same answer?

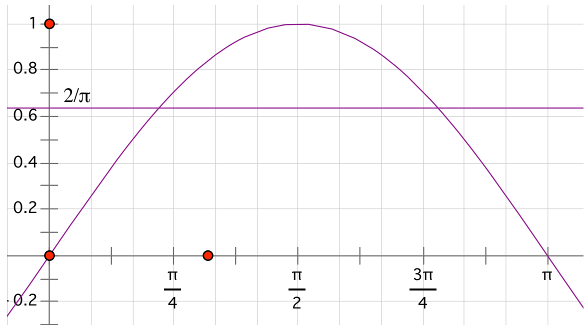
**Definition** If $f(x)$ is a function defined on the interval $[a, b]$, the average value of $f$ on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx.$$

(provided that $f$ is integrable on $[a, b]$.)

**Example** Find the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

Note that if $f(x) \geq 0$ on the interval $[a, b]$, then $(b-a)f_{ave} = \int_a^b f(x)dx$ = area under the curve $y = f(x)$ on the interval $[a, b]$. Hence a rectangle with base $b - a$ and height $f_{ave}$ has the same area as that under the curve $y = f(x)$ on the interval $[a, b]$.




**Mean Value Theorem for Integrals**

If $f$ is a continuous function on $[a, b]$, then there exists a number $c$ in $[a, b]$ such that

$$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx}$$

that is

$$\boxed{\int_a^b f(x)dx = f(c)(b-a).}$$

**Proof** We can apply the mean value theorem for derivatives to the function

$$F(x) = \int_a^x f(t)dt$$

to get

$$\frac{F(b) - F(a)}{b-a} = F'(c)$$

for some $c$ in the interval $[a, b]$. Hence we get

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

for some $c$ in the interval $[a, b]$.

**Note** We see with $f(x) = \sin x$ above that $c$ is not necessarily unique.

**Example** (a) Find the average value of $f(x) = 3x^2 - 3$ on the interval $[1, 3]$.

(b) Find $c$ such that $f_{ave} = f(c)$.

(c) Sketch the graph of $f$ and a rectangle whose area is the same as the area under the graph of $f$ .

**Old Exam Questions** 1. Find the average value of the function $f(x) = \sin^2 x \cos x$ over $[0, \pi/2]$.

2. The function $f(x) = \sqrt{16 - 2x}$ is continuous on the interval $[0, 8]$. Which number below is its average value on this interval.

$(a)$ 8/3 $(b)$ 64/3 $(c)$ $\frac{8}{3}\sqrt{8}$ $(d)$ 16/3 $(e)$ $-\frac{8}{3}$